# A MODEL OF STOCHASTIC VOLATILITY WITH TIME-DEPENDENT PARAMETERS

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## ABSTRACT

We provide the solutions for the Heston model of stochastic volatility when the parameters of the model are constant and when they are functions of time. In the former case the solution follows immediately from the determination of the Lie point symmetries of the governing $1+1$ evolution partial differential equation. This is not the situation in the latter case, but we are able to infer the essential structure of the required nonlocal symmetry from that of the autonomous problem and hence can present the solution to the nonautonomous problem. As in the case of the standard Black-Scholes problem the presence of time-dependent parameters is not a hindrance to the demonstration of a solution.

**Keywords:** Symmetries; stochastic processes; nonlinear evolution equations
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## 1 Introduction

Recently Sophocleous *et al* [26] provided a solution of the Stein-Stein model for stochastic volatility [28] in terms of an algorithmic process based upon the Lie Theory of infinitesimal transformations and its associated group theory. The solution was provided in two instances. The first was the autonomous problem presented by Benth and Karlsen [5] and the second was a nonautonomous version of the same problem introduced by Kufakunesu [14].

In both cases the symmetry analysis showed that the algebraic structure of the evolution partial differential equation of the model,

$$2u_t + \beta^2 u_{xx} - \beta^2 \left(1-\rho^2\right) u_x^2 + 2\left(m - (\alpha + \xi\beta\rho)\, x\right) u_x + \xi^2 x^2 = 0, \tag{1.1}$$

where the parameters, apart from $m$, could depend upon time, was independent of the nature of the functions of time in the coefficients (apart from the natural properties of differentiability to the necessary orders required by the analysis) provided that

$\rho$ was a constant. When coupled with the terminal conditions[1] $u(T,x) = 0$, there were two symmetries remaining. As (1.1) possessed the maximal number of Lie point symmetries, one of the symmetries was a combination of the symmetries associated with the Weyl-Heisenberg subalgebra of the full symmetry group of (1.1) and the second a combination associated with the $sl(2,R)$ subalgebra. This is a not unusual situation in the case of evolution partial differential equations of maximal or near-maximal symmetry when it comes to problems in Financial Mathematics [8, 9, 13, 17, 20, 23, 25].

When $\rho$ was not a constant, *ie* the coefficients of $u_{xx}$ and $u_x^2$ were not constantly proportional, there was a considerable reduction in the number of Lie point symmetries. The infinite subalgebra, indicating that the equation was in fact a linear equation in disguise, disappeared. Also two elements of the previously existing Weyl-Heisenberg subalgebra disappeared. The single remaining symmetry, $\partial_u$, obvious from the absence of $u$ in (1.1) of the Weyl-Heisenberg subalgebra, and the three elements of $sl(2,R)$ remained provided that there was a constraint between the coefficients of the equation. The constraint did not have the simplicity of $\rho$ being a constant! As it happened, the need for the constraint disappeared when one applied the terminal condition. The remaining three symmetries were sufficient to provide a similarity solution of (1.1) subject to the terminal condition.

The richness of the results resulting from the application of symmetry methods to the Stein-Stein model of stochastic volatility prompts one to look at another model, proposed by Heston [12], in which the constant, $m$, is replaced by $m/x$, *ie*, (1.1) becomes

$$2u_t + \beta^2 u_{xx} - \beta^2\left(1-\rho^2\right)u_x^2 + 2\left(\frac{m}{x} - (\alpha + \xi\beta\rho)\,x\right)u_x + \xi^2 x^2 = 0. \tag{1.2}$$

The terminal condition remains as $u(T,x) = 0$. In [5] $\alpha$, $\beta$, $\rho$ and $\xi$ are taken as constants whereas Kufakunesu [14] takes them to have an explicit dependence upon the time.

Our approach to the analysis of (1.2) and the associated terminal condition is based upon the Lie algebraic analysis of the equation to see if there exists a sufficient number of symmetries so that there is the possibility of the existence of a symmetry of the equation which is compatible with the terminal conditions, $u = 0$ when $t = T$ for all $x$. We observe that this approach has been successful in a number of analyses of evolution partial differential equations which arise in Financial Mathematics; see for example [13, 9, 20, 2, 17, 23, 24, 25, 8]. As the calculation of the Lie symmetries of a differential equation is usually a tediously nonintellectual activity, we make use of one of the symbolic manipulation packages available for the purpose. Our choice is Sym [6, 7, 3], but there are several other packages which should be equally effective.

[1] Although this looks like a single condition, in terms of the symmetry analysis it is two since the variables, $t$, $x$ and $u$, are treated as independent variables. Thus the condition mentioned in the full text is in fact the dual condition, $t = T$ and $u = 0$ for all values of $x$.

In view of the number of parameters in (1.2), be they constants or time-dependent functions, an interactive approach is necessary. For this Sym is well-suited. The same is true of the other two packages of known robustness, those of Alan Head [11] and of Clara Nucci [21, 22].

In §2 we analyse (1.2) as in the model of Benth and Karlsen for its Lie point symmetries and see how they can be applied to obtain the solution of the problem with the terminal conditions. We note that there is an interesting algebraic variation in the results. In §3 we make the analysis with the variation proposed by Kufakunesu. We see that there is a big difference in the analysis and that this constitutes one of the more important aspects of this paper. We conclude in §4 with some general comments and observations.

# 2 The Heston Volatility Model

In the classical Black-Scholes-Merton model we assume volatility is constant over an option contract. It is widely accepted that this is an enormous constraint, when developing more general models. One popular initiative is to include the property of stochastic volatility. The model of stochastic volatility due to Heston is considered to be one of the most agreeable to advance analysis. However, one of the limitations is that the closed-form solution to the pricing formula may thus far only be derived when the associated parameters are constant [12] or piecewise constant [18]. This point was advanced in a recent work by Benhamou *et al* [4], for the pricing of European options for time-dependent parameters in the case that the volatility of volatility is relatively small. Here we find solutions to Heston's model for parameters which are constant or functions of time.

When we apply Sym in interactive mode to (1.2) we find that a symmetry has the form

$$\Gamma = a(t)\partial_t + \left(\tfrac{1}{2}\dot{a}x + b(t)\right)\partial_x + \left\{G(t,x) + \exp\left[\left(1-\rho^2\right)u\right]F(t,x)\right\}\partial_u, \tag{2.1}$$

where $F(t,x)$ is a solution of

$$2F_t + \beta^2 F_{xx} + 2\left[\frac{m}{x} - (\alpha+\beta\xi\rho)x\right]F_x - \left(1-\rho^2\right)\xi^2 x^2 F = 0, \tag{2.2}$$

which means that there exists a linearising transformation for (1.2), and

$$G(t,x) = g(t) - \frac{1}{4\beta^2\left(1-\rho^2\right)}\left[\left(\ddot{a} + 2(\alpha+\beta\xi\rho)\dot{a}\right)x^2 + 4\left((\alpha+\beta\xi\rho)b(t) + \dot{b}\right)x - 4\frac{mb(t)}{x}\right] \tag{2.3}$$

The functions $a(t)$, $b(t)$ and $g(t)$ are required to satisfy the system

$$\dddot{a} - 4K^2\dot{a} = 0 \tag{2.4}$$

$$\ddot{b} - K^2 b = 0 \tag{2.5}$$

$$4\beta^2\left(1-\rho^2\right)\dot{g} = \left(2m+\beta^2\right)\left[\ddot{a} + 2\left(\alpha + \beta\xi\rho\right)\dot{a}\right] \tag{2.6}$$

$$4m\left(\beta^2 - m\right)b(t) = 0, \tag{2.7}$$

where $K^2 = \alpha^2 + 2\alpha\beta\xi\rho + \beta^2\xi^2$.

For the nonce we ignore (2.7). However, we take (2.6) into account and write the relevant part[2] of (2.1) as

$$\begin{aligned}\Gamma &= a(t)\partial_t + \left(\tfrac{1}{2}\dot{a}x + b(t)\right)\partial_x \\ &+ \left\{g(t) - \left(\left(2m+\beta^2\right)\dot{g}x^2 - \frac{4\left((\alpha+\beta\xi\rho)b(t) + \dot{b}\right)}{4\beta^2\left(1-\rho^2\right)}x + \frac{4mb(t)}{4\beta^2\left(1-\rho^2\right)x}\right\}\partial_u.\end{aligned} \tag{2.8}$$

We apply (2.8) to the conditions $t = T$ and $u(T,x) = 0$. Since $x$ is a free variable, we equate coefficients of separate powers to zero and obtain the five conditions

$$a(T) = 0 \tag{2.9}$$

$$g(T) = 0 \tag{2.10}$$

$$\dot{g}(T) = 0 \tag{2.11}$$

$$(\alpha+\beta\xi\rho)b(T) + \dot{b}(T) = 0 \quad \text{and} \tag{2.12}$$

$$b(T) = 0, \tag{2.13}$$

of which the first comes from the condition on time and the remaining four from the condition on $u(T,x)$. It is obvious from (2.12) and (2.13) that we can forget about $b(t)$ since the vanishing of both implies the trivial solution for (2.5).

As a consequence of the above the symmetry has the leaner appearance

$$\Gamma = a(t)\partial_t + \tfrac{1}{2}\dot{a}x\partial_x + \left\{g(t) - \left(\left(2m+\beta^2\right)\dot{g}x^2\right\}\partial_u. \tag{2.14}$$

It is now appropriate to look at the characteristics of $\Gamma$ to see what happens to (1.2). The associated Lagrange's system for the invariants of (2.14) is

$$\frac{\mathrm{d}t}{a(t)} = \frac{\mathrm{d}x}{\frac{1}{2}\dot{a}x} = \frac{\mathrm{d}u}{g(t) - \dfrac{\dot{g}(t)x^2}{2m+\beta^2}} \tag{2.15}$$

and the invariants are

$$v = \frac{x^2}{a} \quad \text{and} \quad w = u - \int\left(\frac{g(t)}{a(t)}\right)\mathrm{d}t + \frac{g(t)x^2}{\left(2m+\beta^2\right)a(t)} \tag{2.16}$$

[2] $F(t,x)\partial_u$ plays the role of a solution symmetry and so is of no relevance in the examination of the effect of the terminal condition.

so that the reduction of order to a second-order ordinary differential equation[3] is achieved by means of the transformation

$$u = f(v) + \int \frac{g(t)}{a(t)} \mathrm{d}t - \frac{x^2 g(t)}{(2m+\beta^2)\, a(t)}. \tag{2.17}$$

Before we go to the reduced equation it is apposite to consider the implications of the differential equations for $g(t)$ and $a(t)$ and the three constraints remaining from the imposition of the terminal condition. Recalling from (2.6) that

$$g(t) = G_0 + \frac{2m+\beta^2}{4\beta^2 (1-\rho^2)} \left[2(\alpha + \beta\xi\rho)a(t) + \dot{a}(t)\right]$$

in taking the differential consequence into account we see that (2.9), (2.10) and (2.11) reduce to

$$\begin{aligned} &a(T) = 0 \\ &G_0 + \frac{2m+\beta^2}{4\beta^2(1-\rho^2)}\dot{a}(T) = 0 \\ &\ddot{a}(T) = 0. \end{aligned}$$

When we substitute of the solution of (2.4), namely

$$a(t) = A_0 + A_1 \exp[2Kt] + A_2 \exp[-2Kt],$$

into these conditions and solve them for $A_0$, $A_1$ and $A_2$, we obtain

$$\begin{aligned} A_0 &= 0 \\ A_1 &= -\frac{G_0\beta^2(1-\rho^2)\exp[-2KT]}{K(2m+\beta^2)} \\ A_2 &= \frac{G_0\beta^2(1-\rho^2)\exp[2KT]}{K(2m+\beta^2)}. \end{aligned}$$

With these values for the parameters in the solution the transformation (2.17) reduces (1.2) to

$$4\beta^2 v f'' - 4\beta^2 v \left(1-\rho^2\right) f'^2 + 2\left(\beta^2 + 2m\right) f' = 0, \tag{2.18}$$

where the prime denotes differentiation with respect to the similarity variable, $v$. We note that (2.18) is linearised by means of the transformation

$$z(v) = \exp\left[-\left(1-\rho^2\right) f(v)\right]$$

[3]In the case of the Stein-Stein model the finite-dimensional part of the algebra comprises two subalgebras, $sl(2,R)$ and the three-dimensional Weyl-Heisenberg algebra, $W$, and one obtains reduction to a first-order equation by means of the second subalgebra [26]. In this case we must use the former subalgebra and a more complicated invariant for the similarity variable.

to

$$4\beta^2 v z'' - 2\left(\beta^2 + 2m\right)\left(1-\rho^2\right) z' = 0,$$

where the prime continues to denote differentiation with respect to $v$. The solution of this equation is

$$z(v) = C_0 + C_1 v^{\nu}, \tag{2.19}$$

where $\nu = 1 + (\beta^2 + 2m)(1-\rho^2)/2\beta^2$ and $C_0$ and $C_1$ are constants of integration.

**Remark:** When one considers (1.2) in general, the transformation

$$F(t,x) \longrightarrow x^{-m/\beta^2} \exp\left[\frac{\alpha + \beta\xi\rho}{2\beta^2}\ \left(x^2 - \tfrac{3}{2}\beta^2 t\right)\right] J(t,x) \tag{2.20}$$

reduces (2.2) to

$$2J_t + \beta^2 J_{xx} + \left[\left(\frac{\alpha^2}{\beta^2} + \xi^2\right) x^2 + \left(\frac{m^2}{\beta^2} - m\right)\frac{1}{x^2}\right] J = 0, \tag{2.21}$$

*ie*, the same transformation works independently of any relationship between $m$ and $\beta$. The particular forms of the source/sink functions, as they would be described in terms of the heat equation, are well known in the literature (*cf* [20, 17]).

However, all of these reductions are of no import for the problem under consideration. The substitution, (2.17), gives an equation which contains only the derivatives of $f(v)$ and the nonhomogeneous terms in the transformation vanish. This indicates that, when the terminal condition is taken into account, the nonhomogeneous terms by themselves satisfy (1.2) and so provide a solution to the problem. Since such a solution is to be unique, the equation in $f(v)$ may be ignored, *ie* we should set $f(v) = 0$. This explains the origin of the rather simple dependence upon $x$ in the solution provided by Benth and Karlsen [5]. Since (2.17) reduces to a function containing a potentially complicated exponent, (2.18), such a simple solution is not to be found through (2.18).

# 3 The Heston Volatility Model with Time-dependent Parameters

When the parameters in (1.2) become functions of time, the analysis proceeds in much the same way as for the time-dependent version of (1.1) except that the generality of the time dependence in the parameters does make the computations rather more complicated. We commence with the equation

$$2u_t + p(t)u_{xx} + q(t)u_x^2 + 2\left(\frac{m(t)}{x} - r(t)x\right)u_x + s(t)x^2 = 0. \tag{3.1}$$

The apparent generality in (3.1) is illusory as one of the time-dependent functions can be removed by a rescaling of time. We choose this function to be $p(t)$ and rewrite (3.1) as

$$2u_t + u_{xx} + q(t)u_x^2 + 2\left(\frac{m(t)}{x} - r(t)x\right)u_x + s(t)x^2 = 0, \tag{3.2}$$

*ie*, we may as write $p(t) = 1$ *ab initio*. Since we are not interested in the case for which (3.2) can be linearised by means of the so-called Cole-Hopf transformation, we insist that $q(t)$ be a not constant function.

In the interactive analysis of the determining equations for (3.2) using Sym to find the Lie point symmetries of (3.2) of the form

$$\Gamma = \xi^1(t,x,u)\partial_t + \xi^2(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

we obtain the following results in succession.

1. $\xi^1 = a(t)$,
2. $\xi^2 = b(t) + \frac{1}{2}\dot{a}x$ and
3. $\eta = G(t,x) - ua\dot{q}q - F(t,x)\exp[-uq]$, where $F$ and $G$ satisfy rather complex conditions.

We further analyse these conditions. The two remaining equations contain terms of $t$ and $x$ times various functions of $u$. From the coefficient of $u\exp[-uq]$ in one of the equations we have that $F(t,x) = 0$. The second equation can now be integrated with respect to $x$ to give

$$G(t,x) = g(t) - \frac{bm}{xq} - \frac{a\dot{m}\log x}{q} + \frac{br + \dot{b}}{q}x + \frac{2a\dot{r} + 2\dot{a}r + \ddot{a}}{4q}x^2. \tag{3.3}$$

There remains but one equation which is really too long for meaningful display. We extract the coefficient of $u$ to obtain

$$-4x^2\dot{a}\dot{q} + \frac{4x^2a\dot{q}^2}{q} - 4x^2a\ddot{q} = 0$$

from which it follows that

$$a(t) = \frac{C_1q(t)}{\dot{q}}. \tag{3.4}$$

What is left of the conditions is an equation involving functions of the time as a polynomial in $x$. We extract the various coefficients. From the coefficient of $x^{-1}$ we have $-4bm\dot{q}^3 + 4bm^2\dot{q}^3$ from which it follows that either $b = 0$ or $m = 1$. The options $m = 0$ and $\dot{q} = 0$ remove us from the model under consideration. If we reject $b = 0$, the coefficient of $x$ immediately makes it mandatory. The next consequence is that

$m$ must be a constant, say $M$. This then leaves us with two equations one of which is a first-order equation for $g$ and the other probably most conveniently regarded as a first-order equation for $s$, *ie* even the existence of the limited symmetries as they are imposes a further constraint on the parametric functions in the original equation. Even without considering that part of the problem we realise that we have at most two symmetries and only one of these has a nonzero coefficient function for the operator $\partial_t$. Consequently there can be no Lie point symmetry compatible with the terminal condition for the problem as presently defined and this is the problem which we have insisted cannot be linearised.

In the absence of a Lie point symmetry one has the option either to seek a nonlocal symmetry for (3.1) or to attempt to determine a suitable generalised symmetry. The problems with the latter are that one must make an Ansatz for the nature of the derivative dependence in the generalised symmetry which in itself is really a guessing game and the computations can be rather gruesome. In this case perhaps more than rather gruesome given the summary of the search for a suitable point symmetry presented above. The former option is not really feasible in the case of an evolution equation about which nothing is known apart from the equation itself[4]. It tends to be rather problematic even when one is dealing with an ordinary differential equation [1, 15].

On the assumption that there does exist a nonlocal symmetry which permits the reduction of (3.1) to an ordinary differential equation it must of necessity be an exponential nonlocal symmetry [10] since the similarity variables come from the invariants of the symmetry itself. The nonlocality in the common exponential multiplier cancels from the associated Lagrange's system and leaves it in a form similar to that of (2.15). Equation (3.1) differs from (1.2) in that the parameters are now unspecified functions of time. One recalls that a similar generalisation of the Black-Scholes Equation made no essential difference to the process of solution apart from some problems which could arise in in the performance of quadratures with respect to time [27]. As a final observation we note that (3.1) is even in $x$. All of this suggests that the nonlocal symmetry resembles that in (2.14) subject to the multiplication by an exponential containing the nonlocal term and without the precise specification of the coefficient functions in terms of their dependence upon time. Given our experience with the autonomous problem we make the Ansatz that the solution is of the form

$$u(t,x) = A_0(t) + A_1(t)x^2. \tag{3.5}$$

The terminal condition, $u(T,x) = 0$, implies that $A_0(T) = 0$ and $A_2(T) = 0$.

When we substitute (3.5) into (3.1) and extract the coefficients of independent powers of $x$, we find that the coefficient functions in (3.5) satisfy the equations

$$\dot{A}_2(t) - 2A_2(t)r(t) + 2A_2(t)^2q(t) + \tfrac{1}{2}s(t) = 0 \quad \text{and} \tag{3.6}$$

[4]There can be a difference if one knows some of the properties – in terms of symmetries – of the equation under investigation [19].

$$\dot{A}_0(t) = -A_2(t)(p(t) + 2m(t)), \tag{3.7}$$

*ie*, the determination of the solution of (3.1) subject to the terminal condition $u(T, x) = 0$ reduces to the solution of the Riccati equation (3.6) and the subsequent evaluation of the quadrature implied by (3.7) with the two requirements that $A_0(T) = 0$ and $A_2(T) = 0$. Note that unlike Kufakunesu [14] [ Lemma 2.7, 57ff ] we are unable to solve (3.6) for general functions $q(t)$, $r(t)$ and $s(t)$. It is interesting to observe that the parametric functions of time separate into two groups. The functions $q$, $r$ and $s$ occur in (3.6) as if $A_2$ were $u_x$.

# 4 Concluding Comments

Unlike the Stein-Stein model for stochastic volatility, which was rich in Lie point symmetries whether it be the autonomous or nonautonomous case, the Heston model loses useful point symmetries in the nonautonomous case. For the autonomous case there are sufficient point symmetries to be able to construct the solution. This solution is rather unusual in that the function of the similarity variable is trivially zero and so does not contribute to the solution of the problem with the given terminal condition. What could be termed the nonhomogeneous part of the reduction of $u(t, x)$ to a function of a single variable provides the solution. Although the nonautonomous version of the equation for the Heston model is somewhat lacking in terms of useful point symmetries and then constraints are imposed upon the parametric functions, some important aspects of the route to the solution for the autonomous equation persist. The persistence of these aspects made it possible to infer a likely candidate for the structure of the solution. Consequently it was not necessary to attempt the daunting task of calculating nonlocal symmetries or generalised symmetries. Admittedly this does remain a challenge.

In Benth and Karlsen [5] and in Kufakunesu [14] there is considerable discussion of the uniqueness of the solution obtained. This is because both treated an evolution partial differential equation of far greater complexity than either (1.2) or (3.1). In the cases of the two models considered here there is no need for such an elaborate discussion since the equations lie within the gamut of the Feynman-Kac Theorem.

The model of Heston [12] has proved to be popular over the last approximately two decades. An important feature of our results is that the parameters of the model can be replaced by arbitrary functions of time without losing the property of elementary integrability. Admittedly the solution of the Riccati equation, (3.6), may not be possible in closed form and the quadrature of (3.7) may prove to be quite daunting. Nevertheless a precise structure for the solution has been presented. Its uniqueness is guaranteed by standard theorems. The rest can safely be left to a numerical code. The inclusion of functions of time which may more accurately mimic the reality of the financial world than a collection of constants without having any real deleterious effect upon one's ability to solve the problem is an important advance in the effort to make faithful models.

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